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Some results concerning Riesz–Bessel transforms of high order

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In the present study, we introduce the sharp function related to the Laplace–Bessel differential operator and investigate its properties on the variable Lebesgue spaces. Moreover, we obtain that Riesz–Bessel transforms of high order are bounded on variable Lebesgue spaces $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.

KEYWORDS

sharp function, singular integral operator, variable Lebesgue space

MSC CLASSIFICATION

42B20, 42B35

1 | INTRODUCTION

Harmonic analysis consists of significant operators such as singular integrals, maximal operators, sharp maximal operators, Riesz potentials, convolution type operators, and approximate identities. The problem of boundedness of these operators and their versions, which are generated by Laplace–Bessel differential operators, is important in harmonic analysis. Over the years, this problem has been discussed by Aliev, Ekincioglu, Gadjiev, Guliyev, Kaya, Kipriyanov, Klyuchantsev, Lyakhov, Serbetci, Stempak, and others [1–13].

Recently, there is a big attention on variable Lebesgue spaces, and several results of harmonic analysis have been obtained by Adamowicz et al. [14], Cruz-Uribe et al. [15, 16], Diening [17, 18], and so forth. It is noteworthy to mention that these spaces are Banach spaces. So, there are many similarities with Lebesgue spaces. Nevertheless, some difficulties are encountered while studying on variable Lebesgue spaces. One of them, generalized translation operator T^y is not continuous on variable Lebesgue spaces when $p(\cdot)$ is not constant, particularly. But it is still possible to overcome these difficulties by taking some regularity conditions on this exponent function. In [19], boundedness of generalized translation operator on variable Lebesgue spaces has been obtained. As mentioned earlier, it is assumed that the function $p(\cdot)$ is log-Hölder continuous locally and at infinity. While these assumptions are sufficient, we prefer to work with a weaker hypothesis, which is the boundedness of B-maximal operator M_γ on variable Lebesgue spaces $L_{p(\cdot),\gamma}$. In [2, 8, 20], it has been obtained that B-maximal operators defined on variable Lebesgue spaces are bounded. These studies and the results obtained therein motivate us to investigate the boundedness of Riesz–Bessel transforms of high order on variable Lebesgue spaces $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. In order to obtain the boundedness, we first examine the behavior of the sharp function on variable Lebesgue spaces $L_{p(\cdot),\gamma}$. Then, using this result, we present that Riesz–Bessel transforms of high order on variable Lebesgue spaces are bounded.

The draft of this study is as follows: In Section 2, we recall some basic concepts and definitions, which are useful for us. In Section 3, we introduce the sharp functions associated with the Laplace–Bessel differential operator, for shortly B-sharp functions. Then under the condition $p(\cdot),\, p'(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$, we provide that $\|f\|_{p(\cdot),\gamma} \le C\|f^\sharp\|_{p(\cdot),\gamma}$ for all $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Finally, in Section 4, by using this inequality and the boundedness of B-maximal operator, we have shown that Riesz-Bessel transforms of high order $R^{(k)}_\gamma$ are bounded on variable Lebesgue spaces $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.

2 | PRELIMINARIES

Now, we pause to collect some basic concepts, notations, and known results, which are beneficial for us.

Let $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$, and $S_+ = \{x \in \mathbb{R}^n_{k,+} : |x| = 1\}$. $B_+(x, r)$ denotes the open ball of radius r with center x, namely, $B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. Let $B_+(0, r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0, r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n, k, \gamma) r^{n+|\gamma|},$$

where $\omega(n, k, \gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$.

We will now introduce the spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and recall the basic properties of it. Let $\mathcal{P}(\mathbb{R}^n_{k,+})$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$. The functions in $\mathcal{P}(\mathbb{R}^n_{k,+})$ are called variable exponent functions and also let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n_{k,+}} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} p(x).$$

Given $p(\cdot)$, the conjugate of the exponent function is as follows:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \; x \in \mathbb{R}^n_{k,+}.$$

The analog of log-Hölder continuity for variable Lebesgue spaces related to the Laplace–Bessel differential operator is defined by the following.

Definition 1. Let us give a function $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$. If there exist constants $C_0, C_\infty > 0$, and p_∞ such that for all $|x - y| \le \frac{1}{2}$, and $x, y \in \mathbb{R}^n_{k,+}$,

$$|p(x) - p(y)| \le \frac{C_0}{-\log|x - y|}, \tag{1}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)} \tag{2}$$

hold, then $p(\cdot)$ is log-Hölder continuous on $\mathbb{R}^n_{k,+}$, where $p_\infty = \lim_{x\to\infty} p(x) > 1$. If (1) and (2) hold for $p(\cdot)$, then it is denoted by $p(\cdot) \in \mathcal{P}^{\log}\left(\mathbb{R}^n_{k,+}\right)$ and $p(\cdot) \in \mathcal{P}^{\log}_\infty\left(\mathbb{R}^n_{k,+}\right)$, respectively.

The space $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is known as the set all of measurable functions f such that for a variable exponent $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$,

$$\|f\|_{L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\gamma}(f/\lambda) \le 1\right\} < \infty,$$

where

$$\rho_{p(\cdot),\gamma} := \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)} (x')^\gamma dx.$$

Notice that variable Lebesgue space $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is a Banach space for $1 < p_- \le p(x) \le p_+ < \infty$.

Corollary 1 ([21]). *Let* $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$. *Then* $C_0^\infty\left(\mathbb{R}^n_{k,+}\right)$ *is dense in* $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.

Remark 1. If $p_- > 1$, p is constant outside some large ball B_+ and $p(\cdot) \in \mathcal{P}^{\log}\left(\mathbb{R}^n_{k,+}\right)$, then $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$ and $p'(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$. Also, if $1 \le s < p_-$ and $0 < t \le 1$, then $p(\cdot)/s$, $(p(\cdot)/t)' \in \mathcal{P}^{\log}\left(\mathbb{R}^n_{k,+}\right)$ and are constant outside B_+. Particularly, $p(\cdot)/s$, $(p(\cdot)/t)' \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$.

The generalized translation operator is defined by

$$T^y f(x) := C_{\gamma,k} \int_0^{\pi} \dots \int_0^{\pi} f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma\left(\frac{\gamma_i+1}{2}\right)\left[\Gamma\left(\frac{\gamma_i}{2}\right)\right]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $1 \leq k \leq n$, and $d\gamma(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i$ [10, 22]. Observe that the generalized translation operator is related to the Laplace–Bessel differential operator, and also let us recall the definition of the Laplace–Bessel differential operator below:

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \; B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \; 1 \leq k \leq n.$$

The B-convolution operator connected with T^y is defined by

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^{\gamma} dy.$$

Given a function $f \in L^{\text{loc}}_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, then the B-maximal operator (see [6]) is as follows:

$$M_\gamma f(x) = \sup_{r>0} |B_+(0,r)|_\gamma^{-1} \int_{B_+(0,r)} T^y |f(x)| (y')^{\gamma} dy.$$

For $B_+ \in \mathbb{R}^n_{k,+}$, we can write

$$M_{\gamma,B_+} f := |B_+(x,r)|_\gamma^{-1} \int_{B_+} T^y |f(x)| (y')^{\gamma} dy.$$

Then one can deduce,

$$M_\gamma f := \sup_{B_+(x,r)} M_{\gamma,B_+} f,$$

by taking supremum over all balls centered at x.

In order to define the grand maximal function $\mathcal{M}_{\mathcal{F}}$, let us first recall some basic concepts. The space of the restrictions of the test functions of Schwartz space of rapidly decreasing functions to $\mathbb{R}^n_{k,+}$ will be denoted by $\mathcal{S}_+$, that is,

$$\|f\|_{\alpha,\beta,\gamma} := \sup |x^{\alpha} D_\gamma^{\beta} f|,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha, \beta \in \mathbb{N}_0^n$, and

$$D_\gamma^{\beta} = D_{x'}^{\beta'} B_i^{\beta_i} = D_1^{\beta_1} \dots D_{k-1}^{\beta_{k-1}} B_i^{\beta_i} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \dots \frac{\partial^{\beta_{k-1}}}{\partial x_{k-1}^{\beta_{k-1}}} B_i^{\beta_i}.$$

We denote the finite collection of seminorms on the Schwartz space $\mathcal{S}_+$ of rapidly decreasing functions by $\mathcal{F} := \{\|\cdot\|_{\alpha_i,\beta_i,\gamma}\}$. Let $\mathcal{S}_{\mathcal{F},\gamma}$ be the following subset:

$$\mathcal{S}_{\mathcal{F},\gamma} := \{\Phi \in \mathcal{S}_+ : \|\Phi\|_{\alpha,\beta,\gamma} \leq 1 \text{ for all } \|\cdot\|_{\alpha,\beta,\gamma} \in \mathcal{F}\}.$$

Define $M_{\Phi,\gamma} f(x) := \sup_{t>0} |f * \Phi_t(x)|$ for $\Phi \in L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $\Phi_t(x) := t^{-n}\Phi(x/t)$. If $\Phi \in L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is radial, radially decreasing, and $\Phi \geq 0$, then

$$|M_{\Phi,\gamma} f(x)| \leq \|\Phi\|_{1,\gamma} M_\gamma f(x), \tag{3}$$

for all $f \in L^{\text{loc}}_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. One can easily observe that $\mathcal{M}_{\mathcal{F},\gamma} := \sup_{\Phi \in \mathcal{S}_{\mathcal{F},\gamma}} M_{\Phi,\gamma} f(x)$.

It is simple to obtain that if $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$ and $1 < s < \infty$, then $sp(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$, and

$$\begin{aligned}\|M_\gamma f\|_{sp(\cdot),\gamma} \leq& \left\|M_\gamma(|f|^s)^{\frac{1}{s}}\right\|_{sp(\cdot),\gamma} = \|M_\gamma(|f|^s)\|^{\frac{1}{s}}_{p(\cdot),\gamma} \\ \leq& C\,\||f|^s\|^{\frac{1}{s}}_{p(\cdot),\gamma} = C\|f\|_{sp(\cdot),\gamma}.\end{aligned} \tag{4}$$

3 | *B*-SHARP FUNCTIONS ON VARIABLE LEBESGUE SPACES

This section is devoted to investigate the behavior of the sharp function connected with the Laplace–Bessel differential operator on variable Lebesgue spaces. For this aim, we first consider the following maximal operators and sharp function connected with the following:

For given $f \in L^{\mathrm{loc}}_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right) \cap L^{\mathrm{loc}}_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, and let $0 < p < \infty$, $B_+ \in \mathbb{R}^n_{k,+}$, then

$$\begin{aligned} f_{B_+} &:= |B_+(x,r)|^{-1}_\gamma \int_{B_+} T^y f(x)(y')^\gamma dy, \\ M_{s,B_+,\gamma} &:= \left(|B_+(x,r)|^{-1}_\gamma \int_{B_+} T^y|f(x)|^s(y')^\gamma dy\right)^{\frac{1}{s}}, \\ M^\sharp_{s,B_+,\gamma} &:= \left(|B_+(x,r)|^{-1}_\gamma \int_{B_+} |T^y f(x) - f_{B_+}|^s(y')^\gamma dy\right)^{\frac{1}{s}}, \\ M_{s,\gamma}f(x) &:= \sup_{B_+(x,r)} M_{s,B_+,\gamma}f, \\ M^\sharp_{s,\gamma}f(x) &:= \sup_{B_+(x,r)} M^\sharp_{s,B_+,\gamma}f. \end{aligned}$$

Also, the sharp function $f^\sharp$ is defined by $f^\sharp := M^\sharp_{1,\gamma}f$. Obviously, $f \in BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$ if and only if $f^\sharp \in L_{\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Moreover, it is easy to see that $M_{1,\gamma}f = M_\gamma f$ and for all $p_1 \leq p_2$, $M_{p_1,\gamma}f \leq M_{p_2,\gamma}f$, and $M^\sharp_{p_1,\gamma}f \leq M^\sharp_{p_2,\gamma}f$ by Jensen's inequality.

Lemma 1. *Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$. Then, $f \mapsto f^\sharp$ is continuous on the variable Lebesgue spaces.*

Proof. By using $|f^\sharp| \leq 2M_\gamma f$ and the boundedness of the *B*-maximal operators M_γ on $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ variable Lebesgue spaces (see [2, 8]), the desired result is obtained. □

Lemma 2. *For a given bounded exponent function $p(\cdot)$ such that $p_- < 1$, it is known that Hardy space $H_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is dense in the variable Lebesgue spaces $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.*

The above lemma is an analog of Lemma 3.2 in [21].

Lemma 3. *Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$ and $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then there exist $f_n \in C^\infty_0\left(\mathbb{R}^n_{k,+}\right)$ such that $f_n \to f$, and $f^\sharp_n \to f^\sharp$ on $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.*

Proof. From Corollary 1, since $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$, there exists $f_n \in C^\infty_0\left(\mathbb{R}^n_{k,+}\right)$ with $\|f - f_n\|_{p(\cdot),\gamma} \to 0$. Moreover, since $g^\sharp \leq M_\gamma g$ for all $g \in L^{\mathrm{loc}}_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, and $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$, we get

$$\left\|f^\sharp_n\right\|_{p(\cdot),\gamma} \leq \|M_\gamma f_n\|_{p(\cdot),\gamma} \leq \|f_n\|_{p(\cdot),\gamma}.$$

Therefore, $f_n^\sharp \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. In the same way, $f^\sharp \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Hence, we obtain

$$\begin{aligned}\left\|f^\sharp - f_n^\sharp\right\|_{p(\cdot),\gamma} &\le \|(f - f_n)^\sharp\|_{p(\cdot),\gamma} \le 2\|M_\gamma(f - f_n)\|_{p(\cdot),\gamma} \\ &\le C\|f - f_n\|_{p(\cdot),\gamma} \to 0.\end{aligned}$$

This completes the proof. □

In a similar technique in [21], we can give the following:

Proposition 1. *If* $f \in L_{\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *and* $g \in H_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *then*

$$\left|\int_{\mathbb{R}^n_{k,+}} f(x)g(x)\,dx\right| \le C\int_{\mathbb{R}^n_{k,+}} f^\sharp(x)\mathcal{M}_{\mathcal{F},\gamma}g(x)\,dx$$

holds.

Lemma 4. *Let* $p(\cdot), p'(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$ *with* $1 < p_- \le p_+ < \infty$. *Let* $\mathcal{F}$ *denote any finite collection of seminorm on* $\mathcal{S}_+$, *such that there exists a radial, radially decreasing function* $\Phi_0 \in L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *with* $\Phi_0 \ge 0$ *and* $|\Phi(x)| \le \Phi_0(x)$ *for all* $\Phi \in \mathcal{S}_{\mathcal{F},\gamma}$ *and all* $x \in \mathbb{R}^n_{k,+}$. *Then,*

$$\begin{aligned}\left|\int_{\mathbb{R}^n_{k,+}} f(x)g(x)\,dx\right| &\le C\int_{\mathbb{R}^n_{k,+}} f^\sharp(x)\mathcal{M}_{\mathcal{F},\gamma}g(x)\,dx \\ &\le \|\Phi_0\|_{1,\gamma}\int_{\mathbb{R}^n_{k,+}} f^\sharp(x)M_\gamma g(x)\,dx,\end{aligned}$$

for all $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and* $g \in L_{p'(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$.

Proof. Let $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $g \in L_{p'(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. By Lemma 2 and Lemma 3, there exist sequences $f_n \in C_0^\infty\left(\mathbb{R}^n_{k,+}\right)$ and $g_n \in H_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ such that $f_n \to f$, $f_n^\sharp \to f^\sharp$ on $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$, and $g_n \to g$ on $L_{p'(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. For all $\Phi \in \mathcal{S}_{\mathcal{F},\gamma}$ since $|\Phi| \le \Phi_0$, by (3), we have

$$\begin{aligned}\mathcal{M}_{\mathcal{F},\gamma} = \sup_{\Phi\in\mathcal{F}} M_{\Phi,\gamma}h &\le \sup_{\Phi\in\mathcal{F}} M_{|\Phi|,\gamma}(|h|) \\ &\le M_{\Phi_0,\gamma}(|h|) \le \|\Phi_0\|_{1,\gamma}M_\gamma h,\end{aligned} \tag{5}$$

for all $h \in L^{\mathrm{loc}}_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. We take

$$\mathcal{F} := \left\{\|\cdot\|_{\alpha,\beta,\gamma} : \alpha, \beta \in \mathbb{N}_0^n,\ |\alpha| \le n+1,\ \beta = (0, \dots, 0)\right\},$$

and $\Phi_0(x) := c(1 + |x|^2)^{-\frac{n+1}{2}}$ for suitable $c > 0$. Thus,

$$\begin{aligned}\left\|\mathcal{M}_{\mathcal{F},\gamma}(g - g_n)\right\|_{p'(\cdot),\gamma} &\le \|\Phi_0\|_{1,\gamma}\left\|M_\gamma(g - g_n)\right\|_{p'(\cdot),\gamma} \\ &\le C\|\Phi_0\|_{1,\gamma}\|g - g_n\|_{p'(\cdot),\gamma} \to 0.\end{aligned} \tag{6}$$

By Proposition 1, we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n_{k,+}} f(x)g(x)\,dx\right| &= \lim_{n\to\infty}\left|\int_{\mathbb{R}^n_{k,+}} f_n(x)g_n(x)dx\right| \\ &\le \lim_{n\to\infty} C\left|\int_{\mathbb{R}^n_{k,+}} f_n^\sharp(x)\mathcal{M}_{\mathcal{F},\gamma}g_n(x)\,dx\right| \\ &= C\left|\int_{\mathbb{R}^n_{k,+}} f^\sharp(x)\mathcal{M}_{\mathcal{F},\gamma}g(x)\,dx\right|.\end{aligned}$$

Hence, the proof is completed. ☐

Theorem 1. *Let* $p(\cdot),\, p'(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$*, and* $1 < p_- \le p_+ < \infty$*. Then*

$$\|f\|_{p(\cdot),\gamma} \le C\|f^\sharp\|_{p(\cdot),\gamma},\ \ f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right).$$

Proof. Let $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $g \in L_{p'(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$. By Lemma 4, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n_{k,+}} f(x)g(x)\,dx\right| &\le C\left\|f^\sharp\right\|_{p(\cdot),\gamma}\left\|M_\gamma g\right\|_{p'(\cdot),\gamma}\\ &\le C\left\|f^\sharp\right\|_{p(\cdot),\gamma}\|g\|_{p'(\cdot),\gamma}.\end{aligned}$$

Since $\left(L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)\right)' \cong L_{p'(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$, we get

$$\|f\|_{L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)} \le C\|f^\sharp\|_{L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)}.$$

Hence, the proof is completed. ☐

4 | THE RIESZ–BESSEL TRANSFORMS OF HIGH ORDER

The Riesz–Bessel transforms of high order are defined by

$$\begin{aligned}R^{(k)}_\gamma f(x) &= C_{k,\gamma}\left[p.v\left(\frac{P_k(y)}{|y|^{n+k+|\gamma|}} \otimes f\right)\right](x)\\ &= p.v.C_{k,\gamma}\int_{\mathbb{R}^n_{k,+}} \frac{P_k(y)}{|y|^{n+k+|\gamma|}}\left[T^y f(x)\right](y')^\gamma dy\\ &\equiv C_{k,\gamma}\lim_{\varepsilon\to 0}\int_{\{y\in\mathbb{R}^n_{k,+}\,:\,|y|>\varepsilon\}} \frac{P_k(y)}{|y|^{n+k+|\gamma|}}\left[T^y f(x)\right](y')^\gamma dy \equiv \lim_{\varepsilon\to 0} R^{(k)}_{\gamma,\varepsilon} f(x),\end{aligned} \tag{7}$$

where $C_{k,\gamma} = 2^{(n+|\gamma|)/2}\Gamma((n+|\gamma|)/2)\left[\Gamma(k/2)\right]^{-1}$, $k = 1, 2, \dots n$ [23]. Note that $P_k(x) = P_k(x_1^2, \dots, x_k^2, x_{k+1}, \dots, x_n)$ is a homogeneous polynomial with order k, $\Delta_B P_k = 0$ and satisfying the cancellation condition

$$\int_{S_{k,+}} P_k(\theta)(\theta')^\gamma d\theta = 0 \quad \text{and} \quad \sup_{\theta\in S_+}|P_k(\theta)| = M < \infty, \tag{8}$$

where the characteristic $P_k(\theta)$ belongs to the hemisphere $S_{k,+}$ and $\theta = \frac{y}{|y|}$. We denote by $K(y) = \frac{\Omega_k(y)}{|y|^{n+|\gamma|}}$, $\Omega_k(y) = C_{k,\gamma}\frac{P_k(y)}{|y|^k}$ the kernel of the high-order Riesz–Bessel transforms. One can easily prove that $|T^y f(x) - f(x)| \le c(x)|y|$ holds for Schwartz test functions $f(x)$, by using standard technique.

Lemma 5 ([24])**.** *Let* $0 < \alpha < n + |\gamma|$*. Then,*

$$\left|T^y|x|^{\alpha-n-|\gamma|} - |y|^{\alpha-n-|\gamma|}\right| \le 2^{n+|\gamma|-\alpha+1}|y|^{\alpha-n-|\gamma|-1}|x|$$

holds, for $2|x| \le |y|$*.*

Proposition 2. *Let* $0 < s < 1$*. Then,*

$$\left(M^\sharp_{1,\gamma}(|R^{(k)}_\gamma f(x)|^s)\right)^{\frac{1}{s}}(x) \le CM_\gamma f(x)$$

for all $f \in C_0^\infty\left(\mathbb{R}^n_{k,+}\right)$*, and* $x \in \mathbb{R}^n_{k,+}$*.*

Proof. To prove this, we will show that for each $0 < s < 1$, each ball $B_+ = B_+(x_0, r)$ and for some constant $c = c_{B_+}$, there exists $C > 0$ such that

$$\left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left||T^yR_\gamma^{(k)}f|^s-|c|^s\right|(y')^\gamma dy\right)^{\frac{1}{s}}\le CM_\gamma f(x).$$

Let $f = f_1 + f_2$, where $f_1 = f\chi_{B_+(x_0,r)}$. By using $(R_\gamma^{(k)}f)_{B_+} = \frac{1}{|B_+|_\gamma}\int_{B_+}T^yR_\gamma^{(k)}f(x)(y')^\gamma dy$, we pick $c = (R_\gamma^{(k)}f_2)_{B_+}$. Now, since $||u|^s - |v|^s| \le |u - v|^s$ for $0 < s < 1$ and Riesz–Bessel transforms of high order $R_\gamma^{(k)}$ is linear, we obtain

$$\begin{aligned}
\left(M^\sharp_{1,\gamma}(|R_\gamma^{(k)}f|^s)\right)^{\frac{1}{s}}(x) :&= \left(\sup_{B_+(x)}M^\sharp_{1,B_+(x),\gamma}|R_\gamma^{(k)}f(x)|^s\right)^{\frac{1}{s}}\\
&= \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left||T^yR_\gamma^{(k)}f(x)|^s-|c|^s\right|(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left||R_\gamma^{(k)}(T^yf(x))|^s-|c|^s\right|(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^yf(x))-c\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^y(f_1+f_2)(x))-c\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^yf_1(x))+R_\gamma^{(k)}(T^yf_2(x))-c\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le C\left[\left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^yf_1(x))\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\right.\\
&\qquad\left.+\left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^yf_2(x))-c\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\right]\\
&\le C(I_1+I_2).
\end{aligned}$$

Since $R_\gamma^{(k)}f : L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right) \to L_{1,\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $0 < s < 1$, Bessel-type Kolmogorov inequality for Riesz–Bessel transforms of high order [3] yields

$$\begin{aligned}
I_1 &= \left(\frac{1}{|B_+|_\gamma}\int_{B_+}\left|R_\gamma^{(k)}(T^yf_1(x))\right|^s(y')^\gamma dy\right)^{\frac{1}{s}}\\
&\le \frac{1}{|B_+|_\gamma}|B_+|_\gamma^{1-s}\left[\left(\int_{B_+}|T^yf_1(x)|(y')^\gamma dy\right)^s\right]^{\frac{1}{s}}\\
&\le |B_+|_\gamma^{1-s}\frac{1}{|B_+|_\gamma}\int_{B_+}T^y|f_1(x)|(y')^\gamma dy\\
&\le C\frac{1}{|B_+|_\gamma}\int_{B_+}T^y|f_1(x)|(y')^\gamma dy\\
&\le C\frac{1}{|B_+|_\gamma}\int_{B_+}T^y|f(x)|(y')^\gamma dy\\
&\le CM_\gamma f(x_0).
\end{aligned}$$

To take care of I_2, we consider Jensen's inequality together with Fubini's theorem and then one can obtain that

$$
\begin{aligned}
I_2 &= |B_+|_\gamma^{-1} \int_{B_+} \left| T^y \left(R_\gamma^{(k)} f_2(x) \right) - (R_\gamma^{(k)} f_2)_{B_+} \right| (y')^\gamma dy \\
&= |B_+|_\gamma^{-1} \int_{B_+} \left| T^y \left(R_\gamma^{(k)} f_2(x) \right) - |B_+|_\gamma^{-1} \int_{B_+} T^y \left(R_\gamma^{(k)} f_2(x) \right) (z')^\gamma dz \right| (y')^\gamma dy \\
&\le |B_+|_\gamma^{-1} \int_{B_+} |B_+|_\gamma^{-1} \int_{B_+} \left| R_\gamma^{(k)} (T^y f_2(x)) - R_\gamma^{(k)} f_2(y) \right| (z')^\gamma dz (y')^\gamma dy \\
&\le |B_+|_\gamma^{-2} \int_{B_+} \int_{B_+} \left| \left(\int_{\mathbb{R}^n_{k,+}} K(\tau) T^\tau (T^y f_2(x)) (\tau')^\gamma d\tau - \int_{\mathbb{R}^n_{k,+}} K(\tau) T^\tau f_2(x) (\tau')^\gamma d\tau \right) \right| (z')^\gamma dz (y')^\gamma dy \\
&\le |B_+|_\gamma^{-2} \int_{B_+} \int_{B_+} \left| \left(\int_{\mathbb{R}^n_{k,+}} K(\tau) T^y T^\tau f_2(x) (\tau')^\gamma d\tau - \int_{\mathbb{R}^n_{k,+}} K(\tau) T^\tau f_2(x) (\tau')^\gamma d\tau \right) \right| (z')^\gamma dz (y')^\gamma dy \\
&\le |B_+|_\gamma^{-2} \int_{B_+} \int_{B_+} \left| \int_{\mathbb{R}^n_{k,+}} T^y K(\tau) T^\tau f_2(x) (\tau')^\gamma d\tau - \int_{\mathbb{R}^n_{k,+}} K(\tau) T^\tau f_2(x) (\tau')^\gamma d\tau \right| (z')^\gamma dz (y')^\gamma dy \\
&\le |B_+|_\gamma^{-2} \int_{B_+} \int_{B_+} \left| \int_{B(0,2|x|)} \left[T^y K(\tau) - K(\tau) \right] T^\tau f_2(x) (\tau')^\gamma d\tau \right| (z')^\gamma dz (y')^\gamma dy \\
&\le |B_+|_\gamma^{-2} \int_{B_+} (z')^\gamma dz \int_{B_+} \left[\int_{B(0,2|x|)} |T^y K(\tau) - K(\tau)| (y')^\gamma dy \right] T^\tau f_2(x) (\tau')^\gamma d\tau \\
&\le C \frac{1}{|B_+|_\gamma} \int_{B_+} T^\tau |f_2(x)| (\tau')^\gamma d\tau \\
&\le C M_\gamma f(x_0);
\end{aligned}
$$

that is, it is bounded by a multiple of $M_\gamma f(x_0)$. Hence, the proof is completed. □

We now show that Riesz–Bessel transforms of high order are bounded on variable Lebesgue spaces $L_{p(\cdot),\gamma} \left(\mathbb{R}^n_{k,+} \right)$ for $p(\cdot) \in \mathcal{P} \left(\mathbb{R}^n_{k,+} \right)$.

Theorem 2. *Let* $p(\cdot)$ *be a bounded exponent and* $0 < s < 1$ *such that* $p(\cdot), (p(\cdot)/s)' \in \mathcal{P} \left(\mathbb{R}^n_{k,+} \right)$ *and* $1 < p_- \le p_+ < \infty$. *Then* $R_\gamma^{(k)}$ *is bounded on* $L_{p(\cdot),\gamma} \left(\mathbb{R}^n_{k,+} \right)$*; that is, there exists a constant* $C > 0$*, such that*

$$
\|R_\gamma^{(k)} f\|_{L_{p(\cdot),\gamma}} \le C \|f\|_{L_{p(\cdot),\gamma}}.
$$

Proof. By Proposition 2, we have

$$
\left(M_{1,\gamma}^\sharp (|R_\gamma^{(k)} f|^s) \right)^{\frac{1}{s}} (x_0) \le C M_\gamma f(x_0) \tag{9}
$$

for all $f \in C_0^\infty \left(\mathbb{R}^n_{k,+} \right)$, and $x_0 \in \mathbb{R}^n_{k,+}$. By (4), since $p(\cdot) \in \mathcal{P} \left(\mathbb{R}^n_{k,+} \right)$ and $0 < s < 1$, we have $\frac{p(\cdot)}{s} \in \mathcal{P} \left(\mathbb{R}^n_{k,+} \right)$. Hence,

$$
\|R_\gamma^{(k)} f\|_{p(\cdot),\gamma} = \left\| |R_\gamma^{(k)} f|^s \right\|_{\frac{p(\cdot)}{s},\gamma}^{\frac{1}{s}},
$$

for all $f \in C_0^\infty \left(\mathbb{R}^n_{k,+} \right)$. Since $p(\cdot) \in \mathcal{P} \left(\mathbb{R}^n_{k,+} \right)$, by Theorem 1, (4), and (9), we get

$$
\begin{aligned}
\left\| |R_\gamma^{(k)} f|^s \right\|_{\frac{p(\cdot)}{s},\gamma}^{\frac{1}{s}} &\le C \left\| |R_\gamma^{(k)} f^\sharp|^s \right\|_{\frac{p(\cdot)}{s},\gamma}^{\frac{1}{s}} \\
&\le C \left\| M_{1,\gamma}^\sharp \left(|R_\gamma^{(k)} f|^s \right) \right\|_{\frac{p(\cdot)}{s},\gamma}^{\frac{1}{s}} \\
&\le C \left\| M_{1,\gamma}^\sharp \left(|R_\gamma^{(k)} f|^s \right)^{\frac{1}{s}} \right\|_{p(\cdot),\gamma} \\
&\le CK \left\| M_\gamma f \right\|_{p(\cdot),\gamma} \\
&\le CKC_2 \|f\|_{p(\cdot),\gamma}.
\end{aligned}
$$

Since $C_0^\infty\left(\mathbb{R}^n_{k,+}\right)$ is dense in $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ (see Corollary 1), the desired inequality is obtained. Thus, this proves the theorem. □

5 | CONCLUSIONS

Singular integral operators and boundedness of these operators on various function spaces are considerable problems of harmonic analysis. On variable Lebesgue spaces, singular integral operators have been investigated by many mathematicians. On the other hand, B-singular integral operators and Riesz–Bessel transforms of high order have been studied on Lebesgue spaces. These results motivate us to investigate Riesz–Bessel transforms of high order $R_\gamma^{(k)}$ on variable Lebesgue spaces. In this paper, Riesz–Bessel transforms of high order on variable Lebesgue spaces have been studied. To obtain the mapping properties of these transforms, we first introduce the sharp functions associated with the Laplace–Bessel differential operator. Then, we obtain $\|f\|_{p(\cdot),\gamma} \leq C\|f^\sharp\|_{p(\cdot),\gamma}$ for all $f \in L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$, under the condition $p(\cdot),\ p'(\cdot) \in \mathcal{P}\left(\mathbb{R}^n_{k,+}\right)$. Finally, we show that Riesz–Bessel transforms of high order, $R_\gamma^{(k)}$, are bounded on $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$ by using this estimate and $L_{p(\cdot),\gamma}\left(\mathbb{R}^n_{k,+}\right)$-boundedness of the B-maximal operators.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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