

# On the exact solutions, lie symmetry analysis, and conservation laws of Schamel–Korteweg–de Vries equation

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Communicated by G. Franssens

In this work, we study the integrability aspects of the Schamel–Korteweg–de Vries equation that play an important role in studying the effect of electron trapping on the nonlinear interaction of ion-acoustic waves by including a quasi-potential. Lie symmetry analysis together with the simplest equation method and Kudryashov method is used to obtain exact traveling wave solutions for this equation. In addition, conservation laws are constructed using two different techniques, namely, the multiplier method and the new conservation theorem. Using the conservation laws and symmetries of the underlying equation, double reduction and exact solution were also constructed. Copyright © 2017 John Wiley & Sons, Ltd.

**Keywords:** Lie groups; exact solutions; Schamel–Korteweg–de Vries (S-KdV) equation; conservation laws

## 1. Introduction

Nonlinear evolution equations are popularly used as models to describe many important dynamical systems in various fields of science, particularly in fluid mechanics, solid state physics, plasma physics and nonlinear optics, astrophysics, chemical kinematics, chemical chemistry, optical fiber, and geochemistry. In order to understand better the physical phenomena beside further applications in practical life, it is important to seek much more exact solutions. During the past several decades to generate exact solutions and to understand their properties, several important techniques have been developed such as the sine–cosine method [1], variational iteration method [2–4], homotopy perturbation method [5–8], Jacobi elliptic function method [9], first integral method [10], the  $(G'/G)$ –expansion method [11], the Weierstrass elliptic function expansion method [12], the exponential function method [13], the tanh function method [14], the extended tanh function method [15], the simplest equation method [16–18], and the Lie group analysis method [19–24].

In this paper, we consider the Schamel–Korteweg–de Vries (S-KdV) equation [25]

$$u_t + (\alpha u^{\frac{1}{2}} + \beta u)u_x + \delta u_{xxx} = 0, \quad \beta\delta \neq 0, \quad (1)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants. Equation 1 arises in plasma physics in the study of ion acoustic solitons when electron trapping is present, and also, it governs the electrostatic potential for a certain electron distribution in velocity space. In [26], Tagare and Chakraborti showed that (1) has solitary wave solution by applying direct integral method. In [9], by using exp-function method, Lee and Sakthivel gave some exact traveling wave solutions of (1). In addition, a generalized KdV equation is a particular case of (1) that has been studied in a variety of mathematical physics contexts. When  $\beta = 0$ , (1) becomes the Schamel equation

$$u_t + \alpha u^{\frac{1}{2}}u_x + \delta u_{xxx} = 0,$$

when  $\alpha = 0$ , (1) becomes a well-known KdV equation

$$u_t + \beta uu_x + \delta u_{xxx} = 0$$

that has been studied successively by many authors.

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The outline of the paper is as follows. In Section 2, we obtain Lie point symmetries of the S-KdV equation. Exact solutions are also obtained using the Kudryashov method and the simplest equation method. Then in Section 3, we first recall some definitions and theorem on conservation laws. We construct conservation laws underlying equation using two approaches: the multiplier method and the new conservation method. And utilizing conservation laws, double reduction also constructed. Finally, conclusions are presented.

## 2. Lie point symmetries and exact solutions

### 2.1. Lie point symmetries

We assume that the vector field of the form

$$X = \tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u} \tag{2}$$

will generate the symmetry group of (1). Applying the third prolongation  $X^{[3]}$  to (1), we obtain an overdetermined system of linear partial differential equations. Then, solving the this system, one obtains Lie point symmetries of (1) by using SADE (in MAPLE):

$$\begin{aligned} \text{Case 1 } (\alpha, \beta \neq 0) : & \left\{ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \right\} \\ \text{Case 2 } (\beta = 0) : & \left\{ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = -2u \frac{\partial}{\partial u} + 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right\}, \\ \text{Case 2 } (\alpha = 0) : & \left\{ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{1}{u} \frac{\partial}{\partial u} + 2\beta t \frac{\partial}{\partial x}, \quad X_4 = -u \frac{\partial}{\partial u} + 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right\}. \end{aligned} \tag{3}$$

### 2.2. Exact solutions

We consider the transformations  $u(x, t) = v^2(x, t)$ , and we can rewrite (1) as partial differential equation (PDE) of the form:

$$2vv_t + 2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx} = 0. \tag{4}$$

In order to obtain exact solutions of S-KdV (4), we set  $v(x, t) = V(\xi)$ ,  $\xi = kx - wt$ , then (4) converts to

$$-wVV' + k(\alpha V^2 + \beta V^3) V' + k^3 \delta (VV''' + 3V'V'') = 0, \tag{5}$$

where the prime denotes derivative with respect to  $\xi$ . By once time integrating the earlier equation according to  $\xi$ , then we have

$$\frac{-w}{2} V^2 + \frac{k\alpha}{3} V^3 + \frac{k\beta}{4} V^4 + k^3 \delta [(V')^2 + VV''] = 0. \tag{6}$$

#### 2.2.1. The generalized Kudryashov method.

Now, we apply the Kudryashov method to (6) [27]:  
We assume that the exact solutions of (6) can be expressed by a polynomial in  $Q(\xi)$  as follows:

$$V(\xi) = \frac{a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + \dots + a_N Q^N(\xi)}{b_0 + b_1 Q(\xi) + b_2 Q^2(\xi) + \dots + b_M Q^M(\xi)}. \tag{7}$$

To calculate the values of  $N$  and  $M$  in (7) that is the pole order for the general solution of (6), we balance the highest-order derivative term  $VV''$  with the highest-order nonlinear term  $V^4$ , and we obtain

$$4N - 4M = 2N - 2M + 2 \Rightarrow N = M + 1.$$

If we choose  $N = 2$  and  $M = 1$ , then

$$V(\xi) = \frac{a_0 + a_1 Q + a_2 Q^2}{b_0 + b_1 Q}, \tag{8}$$

where  $Q = Q(\xi) = \frac{1}{1 \mp e^\xi}$  that is the solution of the  $Q_\xi = Q^2 - Q$ .

Substituting (8) into (6) provides a polynomial  $R(Q)$  of  $Q$ . Establishing the coefficients of  $R(Q)$  to zero, we obtain a system of algebraic equations. Solving this system by MAPLE, we have six distinct cases. For example, for

$$\begin{aligned} a_2 &= \frac{-4\alpha}{5\beta} b_0, \quad a_1 = \frac{8\alpha}{5\beta} b_0, \quad a_0 = \frac{-4\alpha}{5\beta} b_0, \\ b_1 &= -2b_0, \quad b_0 = b_0, \\ k &= \frac{\alpha}{5\sqrt{-3\beta\delta}}, \quad w = \frac{-16\alpha^3}{375\beta\sqrt{-3\beta\delta}}. \end{aligned} \tag{9}$$

substituting (9) into (8), we obtain the following solution of (6):

$$V(\xi) = \frac{4\alpha(Q(\xi) - 1)^2}{5\beta(2Q(\xi) - 1)},$$

in other words,

$$V(\xi) = \frac{4\alpha \left( \frac{1}{1 \mp e^\xi} - 1 \right)^2}{5\beta \left( \frac{2}{1 \mp e^\xi} - 1 \right)}. \tag{10}$$

Similarly, we give all exact solutions in the Table I and some figures of this solution with arbitrary constants.

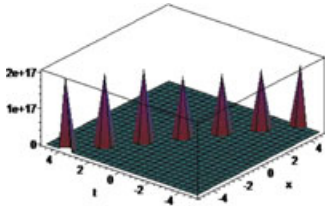


Figure 2.2.1a:  $V_2$  in Table 1 for  $\alpha = 5, \beta = 4, \gamma = -3$

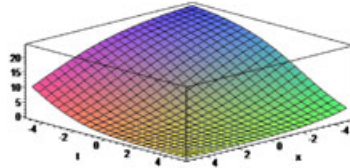


Figure 2.2.1b:  $V_3$  in Table 1 for  $\alpha = 5, \beta = 4, \gamma = -3$

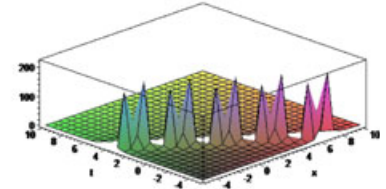


Figure 2.2.1c:  $V_5$  in Table 1 for  $\alpha = 5, \beta = 4, \gamma = -3$

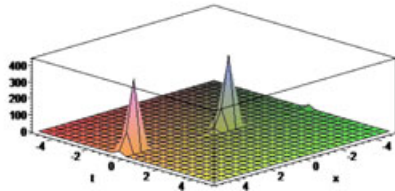


Figure 2.2.1d:  $V_4$  in Table 1 for  $\alpha = 5, \beta = 4, \gamma = -3, k = 1$

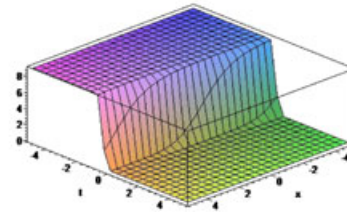


Figure 2.2.1e:  $V_6$  in Table 1 for  $\alpha = -15, \beta = 4, \gamma = -3$

2.2.2. *Simplest equation method.* The simplest equation method was introduced by Kudryashov [16, 18] that will be used to obtain the exact solutions of nonlinear evolution equations.

Let us consider the solution of (6) of the form

$$u(\xi) = \sum_{i=0}^M A_i (G(\xi))^i, \tag{11}$$

where  $G(\xi)$  satisfies the Bernoulli Eq. (13),  $M$  is a positive integer that can be determined by balancing procedure and  $A_0, A_1, \dots, A_M$  are parameters to be determined.

Table I. Exact solutions of (6) using the generalized Kudryashov method (where $Q = Q(\xi) = \frac{1}{1 \mp e^\xi}$ that is the solution of the $Q_\xi = Q^2 - Q$ ).							
$a_2$	$a_1$	$a_0$	$b_1$	$b_0$	$k$	$w$	$V(\xi)$
0	0	$a_0$	0	$b_0$	$k$	$\frac{k(3\beta a_0 + 4\alpha b_0)a_0}{6b_0^2}$	$V_1 = \frac{a_0}{b_0}$
$-\frac{2\alpha b_1}{5\beta}$	0	0	$b_1$	$-\frac{b_1}{2}$	$\frac{\alpha}{5\sqrt{-3\beta\delta}}$	$\frac{-16\alpha^3}{375\beta\sqrt{-3\beta\delta}}$	$V_2 = \frac{-4\alpha Q^2}{5\beta(2Q-1)}$
$a_2$	$a_1$	0	$-\frac{5\beta a_2}{4\alpha}$	$-\frac{5\beta a_1}{4\alpha}$	$\frac{2\alpha}{5\sqrt{-3\beta\delta}}$	$\frac{-32\alpha^3}{375\beta\sqrt{-3\beta\delta}}$	$V_3 = \frac{-4\alpha Q}{\beta}$
$-a_1$	$a_1$	0	$-\frac{a_1\sqrt{-3\beta\delta}}{6k\delta}$	$\frac{a_1(2\alpha + 5k\sqrt{-3\beta\delta})}{60k^2\delta}$	$k$	$4k^3\delta$	$V_4 = \frac{60k^2\delta Q(Q-1)}{-2\alpha + 5k\sqrt{-3\beta\delta}(2Q-1)}$
$-\frac{4\alpha b_0}{5\beta}$	$\frac{8\alpha b_0}{5\beta}$	$-\frac{4\alpha b_0}{5\beta}$	$-2b_0$	$b_0$	$\frac{\alpha}{5\sqrt{-3\beta\delta}}$	$\frac{-16\alpha^3}{375\beta\sqrt{-3\beta\delta}}$	$V_5 = \frac{4\alpha(Q-1)^2}{5\beta(2Q-1)}$
$a_2$	$-\frac{3a_2}{4}$	$-\frac{a_2}{4}$	$\frac{5\beta a_2}{4\alpha}$	$\frac{5\beta a_2}{16\alpha}$	$\frac{2\alpha}{5\sqrt{-3\beta\delta}}$	$\frac{-32\alpha^3}{375\beta\sqrt{-3\beta\delta}}$	$V_{6,7} = \frac{4\alpha}{5\beta}(Q-1)$
$\frac{4\alpha b_1}{5\beta}$	$a_1$	$-a_1 - \frac{4\alpha b_1}{5\beta}$	$b_1$	$\frac{5\beta a_1}{4\alpha} + b_1$	$\frac{2\alpha}{5\sqrt{-3\beta\delta}}$	$\frac{-32\alpha^3}{375\beta\sqrt{-3\beta\delta}}$	

This procedure yields  $M = 1$ , so the solution of (6) is of form

$$u(\xi) = A_0 + A_1 G(\xi). \tag{12}$$

**Bernoulli equation :** The first simplest equation is the Bernoulli equation

$$G'(\xi) = aG(\xi)^2 + bG(\xi), \tag{13}$$

where  $a$  and  $b$  are arbitrary constants. Eq. (13) is a familiar ordinary differential equation (ODE) that posses exact solutions given by elementary functions. The solution can be expressed as

$$G(\xi) = b \left\{ \frac{\cosh [b(\xi + C)] + \sinh [b(\xi + C)]}{1 - a \cosh [b(\xi + C)] - a \sinh [b(\xi + C)]} \right\}, \tag{14}$$

where  $C$  is a constant of integration.

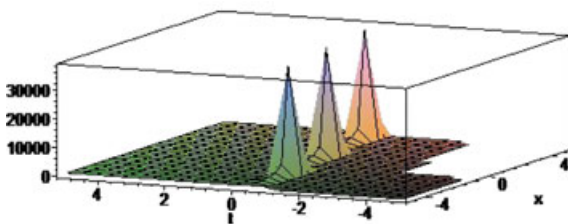
For obtaining an algebraic system of equations in term of  $A_i$ , we substitute (12) into (6) and make use of (13) and then equate the coefficients of the functions  $G^i$  to zero. Solving the resultant system of algebraic equations with the aid of MAPLE, one possible set of values of  $A_i$  ( $i = 0, 1$ ) are

$$\begin{aligned} A_1 &= \mp \frac{6k a \gamma}{\sqrt{-3\beta\gamma}}, & A_{01} &= 0, & w &= 4k^3 b^2 \gamma, & \alpha &= \mp \frac{5kb}{2} \sqrt{-3\beta\gamma} \\ A_1 &= \mp \frac{6k a \gamma}{\sqrt{-3\beta\gamma}}, & A_{02} &= \pm \frac{kb}{2\beta} \sqrt{-3\beta\gamma}, & w &= 4k^3 b^2 \gamma, & \alpha &= \mp \frac{5kb}{2} \sqrt{-3\beta\gamma} \\ A_1 &= \mp \frac{6k a \gamma}{\sqrt{-3\beta\gamma}}, & A_{03} &= \pm \frac{2kb}{\beta} \sqrt{-3\beta\gamma}, & w &= 4k^3 b^2 \gamma, & \alpha &= \mp \frac{5kb}{2} \sqrt{-3\beta\gamma}. \end{aligned}$$

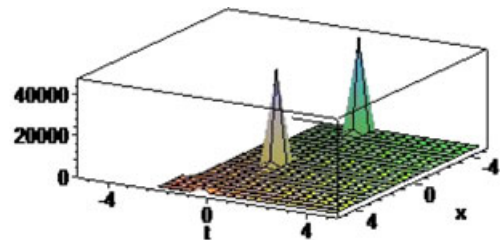
All solutions of (6) are

$$\begin{aligned} V_{11}(\xi) &= \mp \left( \frac{6kab\gamma}{\sqrt{-3\beta\gamma}} \right) \frac{\cosh(b(\xi + c)) + \sinh(b(\xi + c))}{1 - a \cosh(b(\xi + c)) - a \sinh(b(\xi + c))} \\ V_{12}(\xi) &= \pm \frac{kb}{2\beta} \sqrt{-3\beta\gamma} \mp \left( \frac{6kab\gamma}{\sqrt{-3\beta\gamma}} \right) \frac{\cosh(b(\xi + c)) + \sinh(b(\xi + c))}{1 - a \cosh(b(\xi + c)) - a \sinh(b(\xi + c))} \\ V_{13}(\xi) &= \pm \frac{2kb}{\beta} \sqrt{-3\beta\gamma} \mp \left( \frac{6kab\gamma}{\sqrt{-3\beta\gamma}} \right) \frac{\cosh(b(\xi + c)) + \sinh(b(\xi + c))}{1 - a \cosh(b(\xi + c)) - a \sinh(b(\xi + c))} \end{aligned}$$

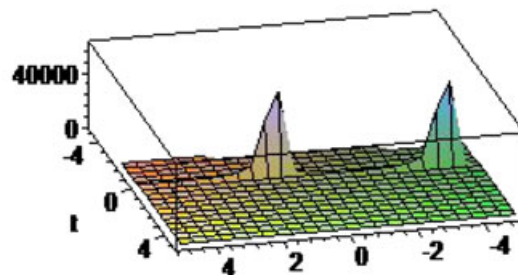
and some figures of this solutions are



**Figure 2.2.2a:**  $V_{11}$   
(for  $\beta = 2, \gamma = 2, k = -1, a = b = c = 1$ )



**Figure 2.2.2b:**  $V_{12}$   
(for  $\beta = 4, \gamma = -3, k = b = 1, a = 2, c = -1$ )



**Figure 2.2.2c:**  $V_{13}$   
(for  $\beta = 16, \gamma = -3, k = a = b = 1, c = -1$ )

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**Riccati equation :** For the Riccati equation that is the second simplest equation

$$G'(\xi) = aG(\xi)^2 + bG(\xi) + d, \tag{15}$$

we shall use the solutions

$$G(\xi) = -\frac{b + \theta \tanh\left(\frac{1}{2}\theta(\xi + C)\right)}{2a} \tag{16}$$

and

$$G(\xi) = -\frac{b + \theta \tanh\left(\frac{1}{2}\theta\xi\right)}{2a} + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)},$$

where  $\theta^2 = b^2 - 4ad > 0$ .

Substituting (12) into (6) and making use of (15) and then the same procedure, we obtain set of values of  $A_i$  ( $i = 0, 1$ ):

$$A_1 = \frac{6ak\gamma}{\sqrt{-3\beta\gamma}}, \quad A_0 = \frac{2\alpha\sqrt{-\beta\gamma} - 5kb\gamma\sqrt{3}\beta}{-5\beta\sqrt{-\beta\gamma}}, \quad w = \frac{-16k\alpha^2}{75\beta}, \quad a = \frac{75k^2b^2\beta\gamma + 4\alpha^2}{300k^2d\beta\gamma}.$$

Solutions of (6) are

$$V_1(\xi) = -\frac{3k\gamma(b + \theta \tanh\left(\frac{1}{2}\theta\xi\right))}{\sqrt{-3\beta\gamma}} + \frac{2\alpha\sqrt{-\beta\gamma} - 5kb\gamma\sqrt{3}\beta}{-5\beta\sqrt{-\beta\gamma}}$$

$$V_2(\xi) = \frac{6ak\gamma}{\sqrt{-3\beta\gamma}} \left( -\frac{b + \theta \tanh\left(\frac{1}{2}\theta\xi\right)}{2a} + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)} \right) + \frac{2\alpha\sqrt{-\beta\gamma} - 5kb\gamma\sqrt{3}\beta}{-5\beta\sqrt{-\beta\gamma}}$$

and some figures of the solutions:

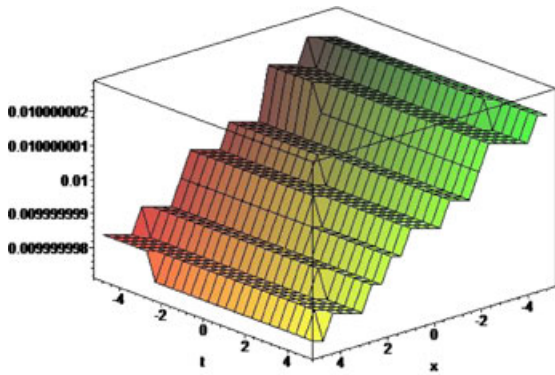


Figure 2.2.2d:  $V_1$

(for  $\alpha = 1, \beta = 4, \gamma = -3, k = 75, b = 4, d = -2$ )

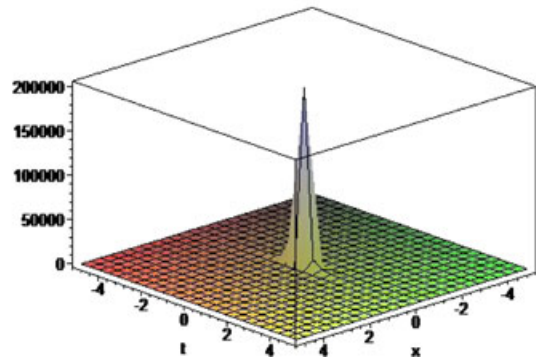


Figure 2.2.2e:  $V_2$

(for  $\alpha = 1, \beta = 4, \gamma = -3, k = 75, b = 4, d = -2$ )

As it can be seen easily that (13) and (15) are equal in case of  $d = 0$ . However, (14) and (16), which are solutions of (13) and (15), respectively, are in coincidence with  $a = -1$ .

### 3. Conservation Laws

In this section, conservation laws will be derived for (4). Firstly, we simply present some notations, definitions, and theorem that will be utilized later (For details [28]).

Consider a  $k$ th-order system of PDEs

$$E_\alpha(x, v, v_{(1)}, \dots, v_{(k)}) = 0, \quad \alpha = 1, \dots, m \tag{17}$$

of  $n$  independent variables  $x = (x^1, x^2, \dots, x^n)$  and  $m$  dependent variables  $v = (v^1, v^2, \dots, v^m)$ . Let  $v_{(1)}, \dots, v_{(k)}$  denote the collections of all first, second, ...,  $k$ th-order partial derivatives. This means that,  $v_i^\alpha = D_i(v^\alpha)$ ,  $v_{ij}^\alpha = D_j D_i(v^\alpha), \dots$ , respectively, where the total derivative operator with respect to  $x^i$  is given by

$$D_i = \frac{\partial}{\partial x^i} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + \dots, \quad i = 1, 2, \dots, n. \tag{18}$$

Now, consider the system of adjoint equations to the system of  $k$ th-order differential Eq. (17) that is defined by Ibragimov [28]

$$E_\alpha^*(x, v, y, \dots, v_{(k)}, y_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{19}$$

where

$$E_\alpha^*(x, v, y, \dots, v_{(k)}, y_{(k)}) = \frac{\delta L(y^\beta E_\beta)}{\delta v^\alpha}, \quad \alpha = 1, \dots, m, \quad v = v(x) \tag{20}$$

and  $y = (y^1, y^2, \dots, y^m)$  are new dependent variables.

(17) is said to be self-adjoint if the substitution of  $y = v$  into the adjoint system (19) yields the same system (17).

If (17) admits the symmetry operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial v^\alpha}, \tag{21}$$

then adjoint system (19) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial v^\alpha} + \eta_*^\alpha \frac{\partial}{\partial y^\alpha}, \quad \eta_*^\alpha = -[\lambda_\beta^\alpha y^\beta + y^\alpha D_i(\xi^i)], \tag{22}$$

where the operator is an extension of (21) to variable  $y^\alpha$  and  $\lambda_\beta^\alpha$  are obtainable from

$$X(E_\alpha) = \lambda_\beta^\alpha E_\beta. \tag{23}$$

**Theorem 1** ([28])

Every Lie point, Lie-Bäcklund and nonlocal symmetry (21) admitted by the system (17) gives rise to a conservation law for the system consisting of the Eq. (17) and the adjoint Eq. (19), where the components  $T^i$  of the conserved vector  $T = (T^1, \dots, T^n)$  are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta v_i^\alpha} + \sum_{k \geq 1} D_{i_1} \dots D_{i_k} (W^\alpha) \frac{\delta L}{\delta v_{i_1 i_2 \dots i_k}^\alpha}, \quad i = 1, \dots, n \tag{24}$$

with Lagrangian give by

$$L = y^\alpha E_\alpha(x, v, \dots, v_{(k)}). \tag{25}$$

In this section, conservation laws will be constructed for (4) by two different methods, namely, the multiplier method and the new conservation method. Exact group-invariant solutions were constructed by double reduction method.

**3.1. The multiplier method**

A multiplier  $\Lambda_\alpha(x, v, v_x, \dots)$  has the property that

$$\Lambda_\alpha E^\alpha = D_i T^i \tag{26}$$

holds identically. Here, we will consider multipliers of third order; that is,  $\Lambda_\alpha = \Lambda_\alpha(x, t, v, v_x, v_{xx}, v_{xxx}, \dots)$ . The right-hand side of (26) is a divergence expression. The determining equation for the multiplier  $\Lambda_\alpha$  is

$$\frac{\delta(\Lambda_\alpha E^\alpha)}{\delta v^\alpha} = 0. \tag{27}$$

Once the multipliers are obtained, the conserved vectors are calculated via homotopy formula [29]. All the multipliers can be calculated by using (27) for which the equation can be expressed as local conservation law [30].

The third-order multiplier for (4) is  $\Lambda(x, t, v, v_x, v_{xx}, v_{xxx})$ , and the corresponding determining equation is

$$\frac{\delta}{\delta v} [\Lambda (2vv_t + 2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx})] = 0. \tag{28}$$

Expanding and then separating (28) with respect to different combinations of derivatives of  $v$  yields overdetermined system for the multipliers. The solution of this system can be expressed as (by using GEM (in MAPLE), see [30])

$$\Lambda = c_1 \left( vv_{xx} + v_x^2 + \frac{\beta}{4\delta} v^4 + \frac{\alpha}{3\delta} v^3 \right) + \frac{1}{2} c_2 v^2 + c_3, \tag{29}$$

where  $c_1, c_2,$  and  $c_3$  are constants. Corresponding to the earlier multiplier, the conserved vectors of (29) are the following:

$$\begin{aligned} T_1^t &= \frac{1}{4}v^4, \\ T_1^x &= \delta v^3 v_{xx} + \frac{\alpha}{5}v^5 + \frac{\beta}{6}v^6, \\ T_2^t &= v^2, \\ T_2^x &= 2\delta (w_{xx} + v_x^2) + \frac{2\alpha}{3}v^3 + \frac{\beta}{2}v^4, \\ T_3^t &= \frac{v^2}{60\delta} (5\beta v^4 + 8\alpha v^3 + 30\delta v v_{xx} + 30\delta v_x^2), \\ T_3^x &= \frac{1}{144\delta} \left( \begin{aligned} &9\beta^2 v^8 + 24\alpha\beta v^7 + 16\alpha^2 v^6 + 72\beta\delta v^5 v_{xx} + 72\beta\delta v^4 v_x^2 \\ &+ 96\alpha\delta v^4 v_{xx} + 96\alpha\delta v^3 v_x^2 + 144\delta^2 v^2 v_{xx}^2 + 288\delta^2 v v_x^2 v_{xx} \\ &+ 144\delta^2 v_x^4 - 72\delta v^3 v_{tx} + 72\delta v^2 v_t v_x \end{aligned} \right). \end{aligned} \tag{30}$$

The multiplier approach gave three local conservation laws for (4).

### 3.2. The new conservation method

In this subsection, we use the new conservation theorem given in [28] and construct conservation laws for (4). The adjoint equation of (4), by invoking (20), is

$$E^*(t, x, v, y, \dots, v_{xxx}, y_{xxx}) = \frac{\delta}{\delta v} [y (2vv_t + 2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx})], \tag{31}$$

where  $y = y(t, x)$  is a new dependent variable. Thus, from (31), we have

$$-2\alpha v^2 y_x - 2\beta v^3 y_x - 2v y_t - 2\delta v y_{xxx} = 0. \tag{32}$$

Clearly, (4) is not self-adjoint from the adjoint Eq. (32). By recalling (25), we obtain the Lagrangian for the system of (4) and (32) as

$$L = y (2vv_t + 2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx}). \tag{33}$$

We give all conserved vectors in Table II.

#### Remark 1

The obtained conservation laws in Table II are nonlocal, which means that they depend upon the solutions of adjoint Eq. (32). However, one can obtain local conservation laws by using the Ibragimov's nonlinearly self-adjointness definition [31]. In [32], the authors shows that the multiplier functions of the considered equation correspond the solution of adjoint equation. For instance, if we consider,  $c_1 = c_3 = 0, c_2 = 1$  in (29) then the multiplier function  $\Lambda = \frac{v^2}{2}$  satisfies the adjoint Eq. (32). Substituting  $y = \frac{v^2}{2}$  in Table II, one can construct local conservation laws of Eq. (4).

### 3.3. Double reduction method

Let  $X$  be any Lie point symmetry and  $T^i$  are the components of conserved vector. If vector field of  $X$  and conserved vector of  $T$  satisfy

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, 2, \tag{34}$$

then  $X$  is associated with  $T$ . We define a new nonlocal variable  $w$  by  $T^t = w_x, T^x = -w_t$ . Taking the similarity variables  $T^r = w_s, T^s = -w_r$ , so that the conservation law is written as

$$D_r T^r + D_s T^s = 0,$$

where ([34], [33])

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)} \tag{35}$$

and

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}. \tag{36}$$

The components  $T^x, T^t$  depend upon  $(x, t, v, v_{(1)}, v_{(2)}, \dots, v_{(q-1)})$ , which means that  $T^s, T^r$  depend upon  $(s, r, \theta, \theta_r, \theta_{rr}, \dots, \theta_{r(q-1)})$  for solutions invariant under  $X$ . Therefore, in the new case divergence condition,  $D_r T^r + D_s T^s = 0$  becomes

$$\frac{\partial T^s}{\partial s} + D_r T^r = 0$$

Table II. Conserved vectors for (4) using the new conservation method.		Case 2	Case 3
	Case 1		
$X_1$	$T_1^1$ $T_1^2$	$T_1^1 = (2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx}) y$ $T_1^2 = (-2\alpha v^2 - v_t 2\beta v^3 v_t - 2\delta v_t v_{xx} - 4\delta v_x v_{tx} - 2\delta v v_{tx}) y$ $+ (2\delta v_t v_x + 2\delta v v_{tx}) y_x - 2\delta v v_t y_{xx}$	$T_1^1$ $T_1^2$
$X_2$	$T_2^1$ $T_2^2$	$T_2^1 = -2v v_x y$ $T_2^2 = 2v v_t y + (2\delta v_x^2 + 2\delta v v_{xx}) y_x - 2\delta v v_x y_{xx}$	$T_2^1$ $T_2^2$
$X_{32}$		$T_{32}^1 = 2(3\alpha t v^2 v_x + 3\beta t v^3 v_x + 9\delta v_x v_{xx} + 3\delta v v_{xxx} - v^2 - x v v_x) y$ $T_{32}^2 = \left( \begin{array}{l} 2x v v_t - 12v v_{xx} - 4\alpha v^3 - 4\beta v^4 - 6\alpha t v^2 v_t \\ -6\beta t v^3 v_t - 6\delta t v v_{xxt} - 12t\delta v_x v_{xt} - 6\delta t v_t v_{xx} - 12\delta v_x^2 \\ + (4\delta v v_x + 2\delta x v_x^2 + 6\delta t v_t v_x + 2\delta x v v_{xx} + 6\delta t w_{xt} + 6\delta v v_x) y_x \\ + (-2\delta x v_x - 6\delta t v_t - 4\delta v) y_{xx} \end{array} \right) y$	
$X_{33}$			$T_{33}^1 = 2y - 4\beta t v v_x y$ $T_{33}^2 = (4\beta t v v_t + 2\alpha v + 2\beta v^2 + 2\delta v^{-1} v_x) y + (2\delta - 4\delta\beta t v_x) y_{xx}$ $+ (-2\delta v^{-1} v_x + 4\delta\beta t v_x^2 + 4\delta\beta t v v_{xx}) y_x$
$X_{43}$			$T_{43}^1 = 3t(2\alpha v^2 v_x + 2\beta v^3 v_x + 6\delta v_x v_{xx} + 2\delta v v_{xxx}) y - 2v^2 y - 2x v v_x y$ $T_{43}^2 = \left( \begin{array}{l} 2x v v_t - 8\delta v_x^2 - 12\delta t v_x v_{xt} - 8\delta v v_{xx} - 6\delta t v v_{xxt} - 2\alpha v^3 \\ -2\beta v^4 - 6\alpha t v^2 v_t - 6\beta t v_t v^3 - 6\delta t v_t v_{xx} \\ + (2\delta x v_x^2 + 6\delta t v_x v_{xt} + 2\delta x v v_{xx} + 6\delta v v_x + 6\delta v_t v_x) y_x \\ + (-2\delta x v v_x - 2\delta v - 6\delta t v_t) y_{xx} \end{array} \right) y$

or equivalently

$$T^r = \int \frac{\partial T^s}{\partial s} dr + f(s).$$

For  $T$  associated with  $X$ , we have  $XT^r = 0$  and  $XT^s = 0$ . Thus,  $T^r$  and  $T^s$  are invariant under  $X$ . This means

$$\frac{\partial}{\partial s} T^r = 0, \frac{\partial}{\partial s} T^s = 0.$$

Overall, the conservation law in canonical coordinates becomes

$$D_r T^r = 0.$$

A PDE  $E = 0$  of order  $q$  with two independent variables, which admits a symmetry  $X$  that is associated with a conserved vector  $T$  is reduced to an ODE of order  $q - 1$ , namely,  $T^r = k_1$ , where  $T^r$  is given by (36) for solutions invariant under  $X$ .

If we consider (4), then (4), admits the symmetry generators  $X_1 = \frac{\partial}{\partial x}$ ,  $X_2 = \frac{\partial}{\partial t}$ , associated with the conservation law

$$D_t \left( \frac{v^2}{2} \right) + D_x \left( \frac{\alpha v^3}{3} + \frac{\beta v^4}{4} + \delta v_x^2 + \delta v v_{xx} \right) = 0. \quad (37)$$

We set  $X = X_1 + cX_2$ . The canonical coordinates of  $X$  are  $s = x$ ,  $r = cx - t$  and  $v$ . Because  $T = (T^r, T^s)$  is associated with  $X$ , we have to find the value of  $T^r$ . Using (36), we obtain the following conserved vector:

$$T^r = k_1 = -\frac{\alpha v^3}{3} - \frac{\beta v^4}{4} - \delta c^2 v_r^2 - \delta c^2 v v_{rr}. \quad (38)$$

We can substitute the variables  $v_r = p$  and  $v_{rr} = p \frac{dp}{dv}$  in (38). After using these variables, (38) reduces to first-order ODE:

$$p(v) = \mp \frac{\sqrt{15\delta(-5\beta v^6 - 8\alpha v^5 - 60k_1 v^2 + 60c_1 \delta c^2)}}{30\delta v c}. \quad (39)$$

We can solve (39) using simple calculations, and the solution gives rise to

$$r + c_2 = \mp \int \frac{30\delta c v}{\sqrt{-15\delta(5\beta v^6 + 8\alpha v^5 + 60k_1 v^2 - 60\delta v^2 c_1)}} dv. \quad (40)$$

## 4. Conclusion

In this work, we studied Lie symmetries, exact solutions, and conservation laws of S-KdV equation. The exact solutions were obtained with the aid of Kudryashov method and simplest equation. Calculated exact solutions were showed in Table I. Using multiplier method, we constructed local conservation laws (30) and utilizing new conservation theorem, we obtain nonlocal conservation laws in Table 2. Then using the double reduction method, we reduced the S-KdV equation to second-order ODE in the canonical variables. Exact group-invariant solutions were constructed by integrating the reduced corresponding ODE.

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