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On the trajectory ruled surfaces of framed base curves in the Euclidean space

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In this paper, we study trajectory ruled surface of a curve with singular points in the Euclidean 3-space as an application of singularity theory of a space curve with singular points. By considering notion of framed curve, we investigate the trajectory ruled surface and give some results about invariants of these surfaces. Then, we give some examples of trajectory ruled surfaces. Moreover, we determine local diffeomorphic image of these surfaces.

KEYWORDS
framed curve, framed surface, singular point, trajectory ruled surface

MSC CLASSIFICATION
14J26; 53A05; 57R45

1 | INTRODUCTION

In the literature, a given surface is called ruled surface provided that it is swept out by a moving line. Ruled surfaces were studied widely in the last century. There are many studies about ruled surfaces and their invariants. One of the special class of the ruled surfaces is the developable ruled surface. Developable ruled surfaces have applications in the study of manufacturing systems, design problems, and computer-aided design. In spatial kinematics, the trajectory of an oriented line embedded in a moving rigid body is generally a ruled surface. So trajectory ruled surfaces are one of the most important topics of ruled surface theory to study spatial mechanism and space kinematic. For more details on ruled surfaces and its applications, see previous studies.[1-6]

In differential geometry, the curve theory is highly important and a classical subject. Generally, geometers consider regular curve, if they work on curve theory. Notwithstanding, there are many articles concerning nonregular curves and singularities of a curve. If a given curve has singular points, the Frenet frame of these curves cannot be constructed. In order to construct the Frenet frame for nonregular curves, we need the notion of framed curve and framed base curve.[7] Tangent vectors of a nonregular curve vanish at singular points, so we will consider a regular spherical curve to construct the Frenet frame. Then, these curves called Frenet-type framed base curve. For more details on framed curves and framed surfaces, see previous studies.[7-11]

In this work, we consider the trajectory ruled surface of a nonregular curve. In order to study these surfaces, we need the notion of framed curves. Trajectory ruled surface whose base curve is Frenet-type framed base curve is investigated. We give some results about invariants of trajectory ruled surfaces. Then, we give examples of these surfaces. We define support function for determining the local diffeomorphic image of these surfaces.

*Trajectory ruled surface of framed base curves.

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2 | PRELIMINARIES

Let $\mathbb{R}^3$ be the three-dimensional Euclidean space with inner product given by

$$\langle u, v\rangle = \sum_{i=1}^{3} u_i v_i,$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Norm of a vector $u \in \mathbb{R}^3$ is defined as $\|u\| = \sqrt{\langle u, u\rangle}$.

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve with moving the Frenet frame $\{T, N, B\}$. Then, Frenet formulas are

$$\begin{aligned}\dot{T} &= \|\dot{\gamma}\|\, \kappa N,\\ \dot{N} &= -\|\dot{\gamma}\|\, \kappa T + \|\dot{\gamma}\|\, \tau B,\\ \dot{B} &= -\|\dot{\gamma}\|\, \tau N,\end{aligned}$$

where κ and τ are curvature and torsion of γ, respectively. Throughout this paper, we consider the curves that have singular points. So we cannot construct Frenet frame with a certain regularity conditions. In order to define the Frenet frame, we apply the theory of framed curves under the following conditions.

Definition 1. A curve $\gamma : I \to \mathbb{R}^3$ is said to be a Frenet-type framed base curve if γ can be written as $\dot{\gamma}(t) = \alpha(t)\mathcal{T}(t)$ where $\mathcal{T} : I \to S^2$ is a regular spherical curve, $\alpha : I \to \mathbb{R}$ is a smooth function, and $t \in I$. Then, $\mathcal{T}(t)$ and $\alpha(t)$ are called a unit tangent vector and a speed function of $\gamma(t)$.[9]

Clearly, t_0 is a singular point of the curve if and only if $\alpha(t)$ vanishes. Then, a unit principal normal vector $\mathcal{N}(t)$ and a unit binormal vector $\mathcal{B}(t)$ can be given as

$$\mathcal{N}(t) = \frac{\dot{T}(t)}{\left\|\dot{T}(t)\right\|} \quad \text{and} \quad \mathcal{B}(t) = \mathcal{T}(t) \times \mathcal{N}(t).$$

An orthonormal frame $\{\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)\}$ is said to be the Frenet-type frame along $\gamma(t)$. Then, the Frenet-Serret–type formula is given by

$$\begin{aligned}\dot{\mathcal{T}}(t) &= \kappa(t)\mathcal{N}(t),\\ \dot{\mathcal{N}}(t) &= -\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t),\\ \dot{\mathcal{B}}(t) &= -\tau(t)\mathcal{N}(t),\end{aligned}$$

where $\kappa(t) = \left\|\dot{\mathcal{T}}(t)\right\|$ is curvature of γ and $\tau(t) = \frac{\det\left(\mathcal{T}(t), \dot{\mathcal{T}}(t), \ddot{\mathcal{T}}(t)\right)}{\left\|\dot{\mathcal{T}}(t)\right\|^2}$ is torsion of γ.[9]

A ruled surface φ is given by

$$\varphi(t, v) = c(t) + v w(t),$$

where c is a base curve and w is a direction vector. The distribution parameter of φ is

$$\delta = \frac{\det\left(c', w, w'\right)}{\|w'\|^2}.$$

A ruled surface is called a developable surface if the tangent plane is constant along with a fixed ruling. Then, the ruled surface is developable if and only if the distribution parameter of ruled surface vanishes. The foot of the common normal between the ruling (generators) is called a striction point. The locus of the striction points is a striction curve, and its parametrization is

$$\beta(s) = c(s) - \frac{\langle c'(s), w'(s)\rangle}{\|w'\|^2} w(s).$$

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3 | TRAJECTORY RULED SURFACES

Let $\gamma : I \to \mathbb{R}^3$ be Frenet-type framed curve. Then, there exists the Frenet-type frame $\left\{\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)\right\}$ along $\gamma(t)$. Let $\mathcal{X}$ be an oriented line in $\mathbb{R}^3$. $\mathcal{X}$ can be written as follows:

$$\mathcal{X}(t) = x_1(t)\mathcal{T}(t) + x_2(t)\mathcal{N}(t) + x_3(t)\mathcal{B}(t), \|\mathcal{X}(t)\| = 1,$$

where $x_1(t), x_2(t)$, and $x_3(t)$ are component functions.

Trajectory ruled surfaces generated by $\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)$ and $\mathcal{X}(t)$ are

$$\begin{aligned} M_1 &: \Phi_1(t,u) = \gamma(t) + u\mathcal{T}(t), \\ M_2 &: \Phi_2(t,v) = \gamma(t) + v\mathcal{N}(t), \\ M_3 &: \Phi_3(t,w) = \gamma(t) + w\mathcal{B}(t), \\ M_4 &: \Phi_4(t,z) = \gamma(t) + z\mathcal{X}(t), \end{aligned}$$

respectively. By a direct computation, the distribution parameters of trajectory ruled surfaces are

$$\begin{aligned} \delta_{\mathcal{T}} &= 0, \\ \delta_{\mathcal{N}} &= \frac{\alpha\tau}{\kappa^2+\tau^2}, \\ \delta_{\mathcal{B}} &= \frac{\alpha}{\tau}, \\ \delta_{\mathcal{X}} &= \frac{\alpha x_2\left(x_2\tau + x_3'\right) - \alpha x_3\left(x_1\kappa + x_2' - x_3\tau\right)}{\left(x_1' - \kappa x_2\right)^2 + \left(\kappa x_1 + x_2' - x_3\tau\right)^2 + \left(x_2\tau + x_3'\right)^2}, \end{aligned}$$

respectively. We can give the following theorems and results without new computation.

Corollary 1. *M_1 surface is developable.*

Corollary 2. *The striction curve of M_1 surface is nonconstant. So M_1 is not a conical surface.*

Corollary 3. *The direction vector of M_1 surface is nonconstant. So M_1 is not a cylindrical surface.*

Corollary 4. *M_2 surface is developable if and only if base curve of M_2 surface is planar (ie, $\tau = 0$).*

Corollary 5. *If $\alpha \neq 0$, then, M_3 surface is nondevelopable.*

Corollary 6. *The striction curve of M_3 surface is the base curve of M_3 surface.*

Corollary 7. *If M_4 surface is developable, then*

$$x_2\left(x_2\tau + x_3'\right) - x_3\left(x_1\kappa + x_2' - x_3\tau\right) = 0.$$

Corollary 8. *Let $\mathcal{X}$ be an oriented line in $\mathcal{N}\mathcal{B}$ plane. If M_4 surface is developable, then*

$$\tau = \left(\frac{x_2}{x_3}\right)' \frac{x_3^2}{\left(x_2^2 + x_3^2\right)}.$$

Corollary 9. *Let $\mathcal{X}$ be an oriented line in $\mathcal{T}\mathcal{B}$ plane. If M_4 surface is developable, then*

$$\frac{\tau}{\kappa} = \frac{x_1}{x_3}.$$

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Corollary 10. *Let $\mathcal{X}$ be an oriented line in $\mathcal{T}\mathcal{N}$ plane. If M_4 surface is developable, then γ is planar.*

Theorem 1. *If $\mathcal{X}$ is a constant oriented line and M_4 surface is developable, then*

$$\frac{\kappa}{\tau} = \frac{x_3^2 - x_2^2}{x_1 x_3}.$$

Theorem 2. *Let $\mathcal{X}$ be a constant oriented line. If M_4 surface is developable, then $\frac{\tau}{\kappa}$ is constant; ie, $\frac{\delta_{\mathcal{B}}}{\delta_{\mathcal{N}}}$ is constant.*

Corollary 11. *γ is framed helix if and only if $\frac{\delta_{\mathcal{B}}}{\delta_{\mathcal{N}}}$ is constant.*

Example 1. Let $\gamma(t) = \left(\cos^3 t, \ \sin^3 t \ , 2\cos^2 t\right)$ be a Frenet-type framed base curve. Then, there exist

$$\alpha(t) = 5\cos t \sin t,$$

and

$$\mathcal{T}(t) = \frac{1}{5}\left(-3\cos t, 3\sin t, -4\right),$$

such that $\gamma'(t) = \alpha(t)\mathcal{T}(t)$. By straightforward computation, $\mathcal{N}(t)$ and $\mathcal{B}(t)$ are

$$\mathcal{N}(t) = \left(\sin t, \cos t, 0\right),$$

and

$$\mathcal{B}(t) = \left(\frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\right).$$

M_1 surface, M_2 surface, and M_3 surface are parametrized by

$$\Phi_1(t,u) = \left(\cos^3 t - \frac{3}{5}u\cos t, \ \sin^3 t + \frac{3}{5}u\sin t, \ \ 2\cos^2 t - \frac{4}{5}u\right),$$
$$\Phi_2(t,v) = \left(\cos^3 t + v\sin t, \ \ \sin^3 t + v\cos t, \ \ 2\cos^2 t\right),$$

and

$$\Phi_3(t,w) = \left(\cos^3 t + \frac{4}{5}w\cos t, \ \ \sin^3 t - \frac{4}{5}w\sin t, 2\cos^2 t - \frac{3}{5}w\right),$$

respectively, and the figures of these surfaces of the domain

$$\begin{cases} 0 \le t \le 2\pi \\ 0 \le u, v, w \le 2 \end{cases},$$

are given in Figures 1 to 3.

M_4 surface is parametrized by

$$\Phi_4(t,z) = \gamma(t) + z\mathcal{X}(t),$$

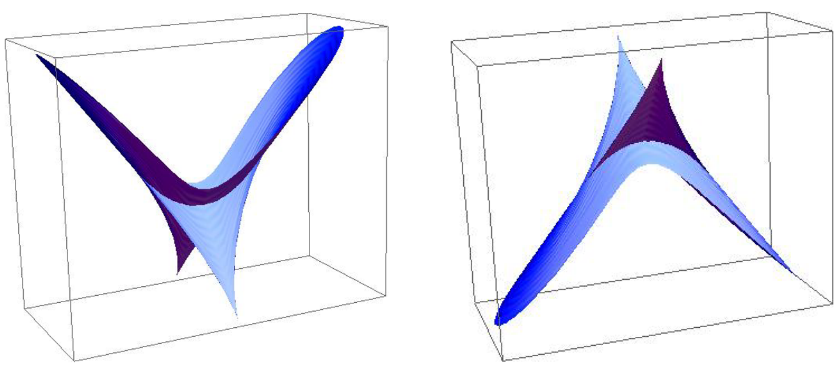

FIGURE 1 M_1 surface [Colour figure can be viewed at wileyonlinelibrary.com]

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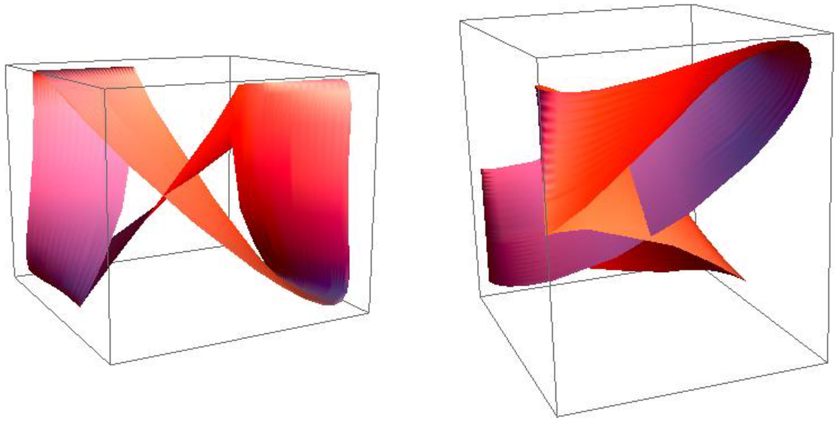

FIGURE 2 M_2 surface [Colour figure can be viewed at wileyonlinelibrary.com]

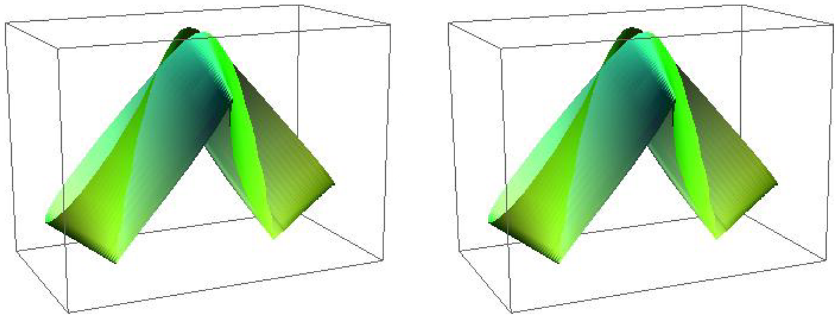

FIGURE 3 M_3 surface [Colour figure can be viewed at wileyonlinelibrary.com]

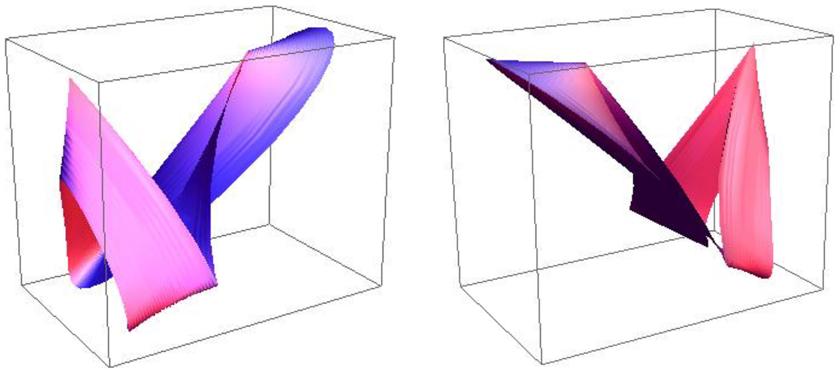

FIGURE 4 M_4 surface [Colour figure can be viewed at wileyonlinelibrary.com]

where $\mathcal{X}(t) = x_1(t)\mathcal{T}(t) + x_2(t)\mathcal{N}(t) + x_3(t)\mathcal{B}(t)$.

If $x_1(t) = \frac{\cos t}{\sqrt{2}}, x_2(t) = \frac{\sin t}{\sqrt{2}}$ and $x_3(t) = \frac{1}{\sqrt{2}}$, then M_4 surface is

$$\Phi_4(t,z) = \left(\cos^3 t + \frac{z(1 + 4\cos t - 4\cos 2t)}{5\sqrt{2}}, \sin^3 t + \frac{4z\sin t(-1 + 2\cos t)}{5\sqrt{2}}, 2\cos^2 t - \frac{z(3 + 4\cos t)}{5\sqrt{2}} \right),$$

and the figure of surface of the domain

$$\begin{cases} 0 \le t \le 2\pi \\ 0 \le z \le 2 \end{cases},$$

is given in Figure 4.

If $x_1(t) = \frac{\cos t}{\sqrt{2}}, x_2(t) = \frac{\cos t}{\sqrt{2}}$, and $x_2(t) = \sin t$, then the figure of M_4 surface of the domain

$$\begin{cases} 0 \le t \le 2\pi \\ 0 \le z \le 2 \end{cases},$$

is given in Figure 5.

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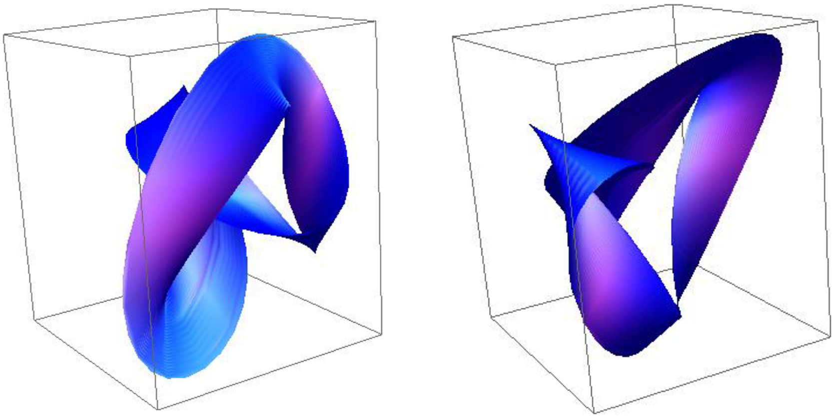

FIGURE 5 M_4 surface [Colour figure can be viewed at wileyonlinelibrary.com]

4 | SUPPORT FUNCTION

In this section, we introduce a support function to study singularities. Let $\gamma : I \to \mathbb{R}^3$ be Frenet-type framed curve. We now define the support function of γ with regard to the unit binormal vector $\mathcal{B}$ on I

$$F : I \times \mathbb{R}^3 \to \mathbb{R},$$

by $F(t,x) = \langle x - \gamma(t), \mathcal{B}(t)\rangle$ and $f_{x_0} = F(t, x_0)$ for any $x_0 \in \mathbb{R}^3$.

Proposition 1. *Let $F(t,x) = \langle x - \gamma(t), \mathcal{B}(t)\rangle$ be the support function of γ and $\tau \neq 0$. Then,*

- i $f_{x_0} = 0$ *if and only if* $x_0 - \gamma = \lambda\mathcal{T} + \mu\mathcal{N}$ *for some* $\lambda, \mu \in \mathbb{R}$.
- ii $f_{x_0} = f'_{x_0} = 0$ *if and only if* $x_0 - \gamma = \lambda\mathcal{T}$ *for some* $\lambda \in \mathbb{R}$.
- iii $f_{x_0} = f'_{x_0} = f''_{x_0} = 0$ *if and only if* $x_0 - \gamma = \lambda\mathcal{T}$ *for some* $\lambda \in \mathbb{R}$ *and* $\kappa = 0$.
- iv $f_{x_0} = f'_{x_0} = f''_{x_0} = f'''_{x_0} = 0$ *if and only if* $x_0 - \gamma = \lambda\mathcal{T}$ *for some* $\lambda \in \mathbb{R}$ *and* $\kappa = 0$ *and* $\kappa' = 0$.

Proof.

- i $f_{x_0} = \langle x_0 - \gamma, \mathcal{B}\rangle = 0 \Leftrightarrow x_0 - \gamma \perp \mathcal{B} \Leftrightarrow x_0 - \gamma = \lambda\mathcal{T} + \mu\mathcal{N}$ for some $\lambda, \mu \in \mathbb{R}$
- ii $f_{x_0} = f'_{x_0} = 0 \Leftrightarrow \langle x_0 - \gamma, \mathcal{B}\rangle = \left\langle x_0 - \gamma, -\tau\mathcal{N}\right\rangle = 0 \Leftrightarrow \mu\tau = 0 \Leftrightarrow \tau \neq 0$ and $\mu = 0 \Leftrightarrow x_0 - \gamma = \lambda\mathcal{T}$, $\lambda \in \mathbb{R}$ and $\tau \neq 0$.
- iii $f_{x_0} = f'_{x_0} = f''_{x_0} = 0 \Leftrightarrow \langle x_0 - \gamma, \mathcal{B}\rangle = \left\langle x_0 - \gamma, -\tau\mathcal{N}\right\rangle = \langle x_0 - \gamma, -\kappa\tau\mathcal{T}\rangle = 0 \Leftrightarrow x_0 - \gamma = \lambda\mathcal{T}$, $\lambda \in \mathbb{R}$ and $\kappa = 0$.
- iv $f_{x_0} = f'_{x_0} = f''_{x_0} = f'''_{x_0} = 0 \Leftrightarrow \langle x_0 - \gamma, \mathcal{B}\rangle = \left\langle x_0 - \gamma, -\tau\mathcal{N}\right\rangle = \left\langle x_0 - \gamma, -\kappa\tau\mathcal{N}\right\rangle = \langle \alpha\mathcal{T}, -\kappa'\tau\mathcal{T}\rangle = 0 \Leftrightarrow x_0 - \gamma = \lambda\mathcal{T}$, $\lambda \in \mathbb{R}$, $\kappa = 0$ and $\kappa' = 0$.

□

In order to determine the local diffeomorphic image of M_1 surface, we need the theorem 6.10 and the definition of the unfolding and A_k singularity in[14] For more details on the singularity theory, see[12-19]

Definition 2. Let $F : \mathbb{R} \times \mathbb{R}^r (t_0, x_0) \to \mathbb{R}$ be a smooth function. Then, F is called r parameter unfolding of f where $F_{x_0}(t) = F(t, x_0)$, $f = F_{x_0}$.

Definition 3. Let $f : \mathbb{R}, t_0 \to \mathbb{R}$ be a smooth function. If $f^{(p)}(t_0) = 0$ for all p with $1 \leq p \leq k$ and $f^{(p+1)}(t_0) \neq 0$, then f is said to have A_k singularity at t_0.

The discriminant set of F is given as follows:

$$\mathcal{D}_F = \left\{ x \in \mathbb{R}^r \,\middle|\, F(s,x) = \frac{\partial F}{\partial s} = 0 \quad \text{at} \quad (t,x) \right\},$$

and by a direct computation, the discriminant set of F is obtained as M_1 surface.

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Definition 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function that has at least k derivatives. k jet of f at $x_0 \in \mathbb{R}^n$ is given by

$$j^{k-1}\left(\frac{\partial F(s,x_0)}{\partial x_i}\right)(s_0) = a_{0i} + a_{1i}s + a_{2i}s^2 + \dots + a_{k-1,i}s^{k-1}.$$

Theorem 3. *Let $(k-1)$ jet be $j^{k-1}\left(\frac{\partial F_i(s,x_0)}{\partial x}\right)(s_0) = a_{0i} + a_{1i}s + a_{2i}s^2 + \dots + a_{k-1,i}s^{k-1}$ for $i = 1, \dots, r$, where $x \in \mathbb{R}^r$ and F is an r parameter unfolding. Then, $F(s,x)$ is versal if and only if the $k \times r$ matrix of coefficients (a_{ji}) has rank k (for $j = 0, \dots, k; i = 1, \dots, r$).*[14]

The notions of the (p)versal unfolding and the versal unfoldings can be found in.[14]

Theorem 4. *Let $F(t,x) = \langle x - \gamma(t), \mathcal{B}(t)\rangle$ be the support function of γ. If $f(t)$ has the A_2 singularity at t_0 and $\tau \neq 0$, then F is the versal unfolding of f_{x_0}.*

Proof. For $x = (x_1, x_2, x_3)$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\mathcal{B}(t) = (b_1(t), b_2(t), b_3(t))$, we can write support function F as

$$F(t,x) = b_1(t)(x_1 - \gamma_1(t)) + b_2(t)(x_2 - \gamma_2(t)) + b_3(t)(x_3 - \gamma_3(t)).$$

By taking derivative of F with respect to x_i, we obtain

$$\frac{\partial F}{\partial x_i} = b_i, i = 1, 2, 3.$$

Let $j^{k-1}\left(\frac{\partial F(t,x_0)}{\partial x_i}\right)(t_0)$ be the $(k-1)$ jet of $\frac{\partial F}{\partial x_i}$ at t_0. Then, we have

$$j^1\left(\frac{\partial F(t,x_0)}{\partial x_i}\right)(t_0) = b_i(t_0) + b_i'(t_0)(t - t_0).$$

Thus, coefficient matrices C is as follows:

$$C = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \end{bmatrix}.$$

By the Frenet-Serret–type formula, we obtain

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\tau & 0 \end{bmatrix}.$$

If $f(s)$ has the A_2 singularity at t_0 and $\tau \neq 0$, then rank $C = 0$. So by Theorem 3 (6.10 in[14]), F is a versal unfolding of f. □

Theorem 5. *Given γ be a Frenet-type framed curve with $\mathcal{T}(t)$ with $\tau \neq 0$ and F be a three-parameter unfolding of f, which has the A_2 singularity at t_0. If F is versal unfolding, then M_1 surface is locally diffeomorphic to the cuspidal edge.*

5 | CONCLUSION

In this paper, by considering notion of framed curve, we studied trajectory ruled surfaces and their local diffeomorphic images. In a similar manner to this paper, ruled surfaces, which generated by the spherical indicatrices of a nonregular curve.

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FINANCIAL DISCLOSURE

None reported.

CONFLICT OF INTERESTS

The authors declare no potential conflict of interests.

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